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ON AN IRREGULAR DIFFERENTIAL GAME*

A.M. TARAS'EV

An example of a differential game is considered and its cost function found. If the cost function is known, the construction of the optimal strategies becomes relatively simple /1/. In the regions where the cost function is differentiable, it satisfies a Hamilton-Jacobi type partial differential equation, i.e. the Isaacs-Bellman /2/ equation which, as a rule, is degenerate. Therefore we have, in position space, so-called singular sets on which the cost function is differentiable, and this creates considerable difficulities in determining this function. Only a small number of cases have been known until now, for which the problem in question has been successfully solved. Although the equations of motion and the payoff function used in the present case are fairly simple, they fully illustrate the difficulties mentioned above.

The results obtained can be of use in studying singular surfaces. The formulas given below can be used to control the algorithms developed for computing the cost of the game. It should also be noted that the possibility, for the case in question, of applying the differential inequalities /3, 4/ to the cost function was tested.

Consider the conflict-controlled system

$$x_1^{\cdot} = x_2 + v,\ x_2^{\cdot} = u;\ |u| \leqslant 1,\ |v| \leqslant 1 \tag{1}$$

in the time interval $I = [0, 2]$. Here x_1, x_2, u, v are scalars, while u and v are the controls of the first and second player.

The quantity $\sigma(x(2))$ represents the payoff, where

$$\sigma(x) = \max\{|x_1|, |x_2|\} \tag{2}$$

The Isaacs-Bellman equations for the system (1) have the form

$$\frac{\partial c}{\partial t} + x_2\frac{\partial c}{\partial x_1} - \left|\frac{\partial c}{\partial x_1}\right| - \left|\frac{\partial c}{\partial x_2}\right| = 0 \tag{3}$$

We have succeeded in constructing the cost function $c(t, x)$ for the differential game in question, in the position space $I \times R^2$. We have found that the cost function is composed of twelve smooth functions.

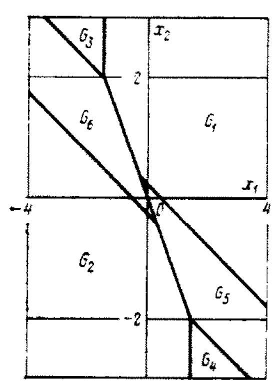

Fig.1

Let us first explain the following circumstance. It may appear that the complexity of the solution obtained is caused by the choice of a "very poor" payoff function. This is not so. For example, when the payoff function chosen is $\sigma(x) = \|x\|$, the cost function is obtained by joining together a smaller number of smooth functions than in the case of the payoff of the type (2). However, we find that constructing these smooth functions is much more complicated than in the case in question. We note that the controlled system of the type (1) was studied earlier**(**Patsko V.S. and Tarasova S.I. Differential game of approach with fixed termination time. Sverdlovsk, 1983.) with another payoff functional.

Let us write the expressions defining the functions mentioned above

$$q_1 = q_1(t, x) = x_1 + (2 - t)x_2 + (2 - t) - {}^1/_2(2 - t)^2$$
$$q_3 = q_3(t, x) = x_2 - (2 - t)$$
$$q_5 = q_5(t, x) = 2 - q_3 - 2(1 - q_1 - q_3)^{1/2}$$
$$q_i = q_i(t, x) = q_{i-1}(t, x);\ i = 2, 4, 6$$

The following six functions are defined implicitly:

$$\Phi_7(t, x, q_7) = 2\ln(1 + (1 - q_1 - q_7)^{1/2}) - 2(1 - q_1 - q_7)^{1/2} - q_3 - q_7 - 2(1 + \ln q_7) = 0$$
$$\Phi_8(t, x, q_8) = 2\ln(1 + (1 - q_1 - q_8)^{1/2}) - 2(1 - q_1 - q_8)^{1/2} - q_3 - q_8 - 2(2q_8 - 1)^{1/2} = 0$$

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$$\Phi_{11}(t, x, \varphi_{11}) = -(1 - \varphi_1 - \varphi_{11}) + {}^1/_4 (\varphi_{11} - \varphi_3 - 2)^2 + (\varphi_{11} - \varphi_3 - 2) \times$$
$$(\ln(\varphi_{11} - \varphi_3 - 2) - \ln(2(\Psi(\varphi_{11}) - 1))) = 0$$
$$\Psi(\varphi) = \exp(\varphi - 1 + (2\varphi - 1)^{3/2})$$
$$\Phi_i(t, x, \varphi_i) = \Phi_{i-1}(t, -x, \varphi_i) = 0; \ i = 8, 10, 12$$

We note that $\partial\Phi_i(t, x, \varphi_i)/\partial\varphi_i < 0$ for all (t, x, φ_i) from the domain of definition of the functions $\Phi_i(t, x, \varphi_i)$ $(i = 7, \ldots, 12)$. The function $\varphi_i(t, x)$ $(i = 7, \ldots, 12)$ is found uniquely from the corresponding equation. This enables us to use the necessary and sufficient conditions /3, 4/ which must be satisfied by the cost function, for confirming the correctness of the results obtained.

The cost function of the differential game (1), (2) is determined as follows. The position space $I \times R^2$ is decomposed* (*An analytic description of this decomposition can be found in the paper by A.M. Taras'ev. On constructing the cost function in an irregular differential game with fixed termination time. Sverdlovsk 1983. Constructions leading to the formulas describing the cost function and their justification can also be found there.) into twelve regions G_i $(i = 1, \ldots, 12)$. Figs.1 and 2 depict the cross-sections of the regions G_i $(i = 1, \ldots, 12)$ at the instants $t_1 = 1, t_2 = 0$ respectively (the right-hand side of Fig.2 depicts the central section in enlarged form).

The cost function of the problem (1), (2) is written in the form

$$c(t, x) = \varphi_i(t, x), \ (t, x) \in G_i \ (i = 1, \ldots, 12) \tag{4}$$

We note that when $t \in [1, 2]$, the cost of the game is identical with the programmed maximum and has a simple analytic description

$$c(t, x) = \begin{cases} \max\{\varphi_1, \varphi_3, \varphi_6\}, & \text{if} \quad \varphi_1 + \varphi_3 \geqslant 0 \\ \max\{\varphi_2, \varphi_4, \varphi_5\}, & \text{if} \quad \varphi_2 + \varphi_4 \geqslant 0 \\ \max\{\varphi_5, \varphi_6\}, & \text{if} \quad \varphi_1 + \varphi_3 \leqslant 0 \ \text{ and } \ \varphi_2 + \varphi_4 \leqslant 0 \end{cases}$$
$$(\varphi_i = \varphi_i(t, x), \ i = 1, \ldots, 6)$$

In the strip $[0, 1] \times R^2$ the cost of the game is not, in general, identical with the programmed maximum and has a more complex structure.

In conclusion we note that the cost function was found using intelligible procedures and the corresponding passages to the limit /5–8/. The final check of the functions obtained in this manner was carried out with help of the necessary and sufficient conditions /3, 4/ which had to be satisfied by the cost function.

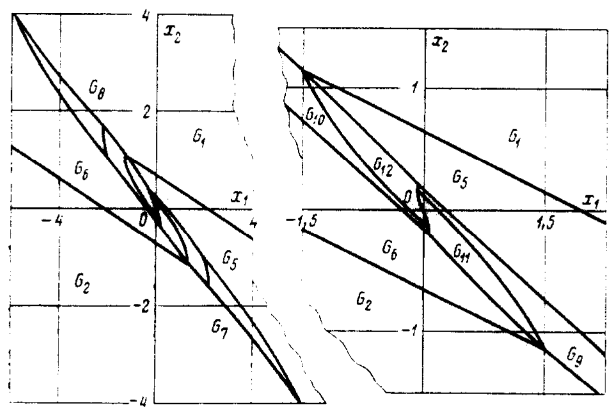

Fig.2

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